A TRICHOTOMY FOR A CLASS OF EQUIVALENCE RELATIONS

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ABSTRACT. Let X_n, $n \in \mathbb{N}$ be a sequence of non-empty sets, $\psi_n : X_n^2 \to \mathbb{R}^+$. We consider the relation $E((X_n, \psi_n)_{n\in\mathbb{N}})$ on $\prod_{n\in\mathbb{N}} X_n$ by $(x, y) \in E((X_n, \psi_n)_{n\in\mathbb{N}}) \Leftrightarrow \sum_{n\in\mathbb{N}} \psi_n(x(n), y(n)) < +\infty$. If $E((X_n, \psi_n)_{n\in\mathbb{N}})$ is a Borel equivalence relation, we show a trichotomy that either $\mathbb{R}^{\mathbb{N}}/\ell_1 \le_B E$, $E_1 \le_B E$, or $E \le_B E_0$.

We also prove that, for a rather general case, $E((X_n, \psi_n)_{n\in\mathbb{N}})$ is an equivalence relation iff it is an ℓ_p-like equivalence relation.

1. INTRODUCTION

A topological space is called a *Polish space* if it is separable and completely metrizable. Let X, Y be Polish spaces and E, F equivalence relations on X, Y respectively. A *Borel reduction* of E to F is a Borel function $\theta : X \to Y$ such that $(x, y) \in E$ iff $(\theta(x), \theta(y)) \in F$, for all $x, y \in X$. We say that E is *Borel reducible* to F, denoted $E \le_B F$, if there is a Borel reduction of E to F. If $E \le_B F$ and $F \le_B E$, we say that E and F are *Borel bireducible* and denote $E \sim_B F$. We refer to [4] and [8] for background on Borel reducibility.

There are several famous dichotomy theorems on Borel reducibility. The first one is the Silver's dichotomy theorem [13].

Theorem 1.1 (Silver)**.** *Let E be a $\mathbf{\Pi}^1_1$ equivalence relation. Then E has either at most countably many or perfectly many equivalence classes, i.e. $E \le_B \mathrm{id}(\mathbb{N})$ or $\mathrm{id}(\mathbb{R}) \le_B E$.*

There are three dichotomy theorems concerning E_0. Before introducing these theorems, we recall definitions of equivalence relations E_0, E_1, E_0^ω.

(a) For $x, y \in 2^{\mathbb{N}}$, $(x, y) \in E_0 \Leftrightarrow \exists m \forall n \ge m(x(n) = y(n))$.
(b) For $x, y \in 2^{\mathbb{N}\times\mathbb{N}}$, $(x, y) \in E_1 \Leftrightarrow \exists m \forall n \ge m \forall k(x(n, k) = y(n, k))$.
(c) For $x, y \in 2^{\mathbb{N}\times\mathbb{N}}$, $(x, y) \in E_0^\omega \Leftrightarrow \forall k \exists m \forall n \ge m(x(n, k) = y(n, k))$.

Theorem 1.2. *Let E be a Borel equivalence relation. Then*

(a) (Harrington-Kechris-Louveau [5]) *either $E \le_B \mathrm{id}(\mathbb{R})$ or $E_0 \le_B E$;*
(b) (Kechris-Louveau [9]) *if $E \le_B E_1$, then $E \le_B E_0$ or $E \sim_B E_1$;*
(c) (Hjorth-Kechris [7]) *if $E \le_B E_0^\omega$, then $E \le_B E_0$ or $E \sim_B E_0^\omega$.*

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Another class of interesting Borel equivalence relations come from classical Banach sequence spaces. Let $p \geq 1$. For $x, y \in \mathbb{R}^{\mathbb{N}}$, $(x, y) \in \mathbb{R}^{\mathbb{N}}/\ell_p \Leftrightarrow x - y \in \ell_p$. It was shown by G. Hjorth [6] that every Borel equivalence relation $E \leq_B \mathbb{R}^{\mathbb{N}}/\ell_1$ is either essentially countable or satisfies $E \sim_B \mathbb{R}^{\mathbb{N}}/\ell_1$. Kanovei asked whether the position of $\mathbb{R}^{\mathbb{N}}/\ell_1$ in the $\leq_B$-structure is similar with E_1 and E_0^ω (see [8], Question 5.7.5).

Question 1.3 (Kanovei)**.** Does every Borel equivalence relation $E \leq \mathbb{R}^{\mathbb{N}}/\ell_1$ satisfy either $E \leq_B E_0$ or $E \sim_B \mathbb{R}^{\mathbb{N}}/\ell_1$?

Two kinds of ℓ_p-like equivalence relations were introduced by T. Mátrai [11] and the author [1]. (1) Let $f : [0, 1] \to \mathbb{R}^+$. For $x, y \in [0, 1]^{\mathbb{N}}$, $(x, y) \in E_f \Leftrightarrow \sum_{n\in\mathbb{N}} f(|x(n) - y(n)|) < +\infty$. (2) Let (X_n, d_n), $n \in \mathbb{N}$ be a sequence of metric spaces, $p \geq 1$. For $(x, y) \in \prod_{n\in\mathbb{N}} X_n$, $(x, y) \in E((X_n, d_n)_{n\in\mathbb{N}}; p) \Leftrightarrow \sum_{n\in\mathbb{N}} d_n(x(n), y(n))^p < +\infty$.

In this paper, we introduce a notion surpassing both (1) and (2). Let X_n, $n \in \mathbb{N}$ be a sequence of non-empty sets, $\psi_n : X_n^2 \to \mathbb{R}^+$. For $x, y \in \prod_{n\in\mathbb{N}} X_n$, $(x, y) \in E((X_n, \psi_n)_{n\in\mathbb{N}}) \Leftrightarrow \sum_{n\in\mathbb{N}} \psi_n(x(n), y(n)) < +\infty$. Though we did not find a natural necessary and sufficient condition that $E((X_n, \psi_n)_{n\in\mathbb{N}})$ be an equivalence relation, we establish the following trichotomy.

Theorem 1.4. *If $E = E((X_n, \psi_n)_{n\in\mathbb{N}})$ is a Borel equivalence relation, then either $\mathbb{R}^{\mathbb{N}}/\ell_1 \leq_B E$, $E_1 \leq_B E$, or $E \leq_B E_0$.*

From this trichotomy, we can see that Kanovei's problem is valid within equivalence relations of the form $E((X_n, \psi_n)_{n\in\mathbb{N}})$.

It was shown by R. Dougherty and G. Hjorth [2] that, for $p, q \geq 1$, $\mathbb{R}^{\mathbb{N}}/\ell_p \leq_B \mathbb{R}^{\mathbb{N}}/\ell_q$ iff $p \leq q$. We complete a different picture for $0 < p \leq 1$ by showing that $\mathbb{R}^{\mathbb{N}}/\ell_p \sim_B \mathbb{R}^{\mathbb{N}}/\ell_1$.

Via a process of metrization, we prove that, for a rather general case, equivalence relations $E((X_n, \psi_n)_{n\in\mathbb{N}})$ coincide with ℓ_p-like equivalence relations $E((X_n, d_n)_{n\in\mathbb{N}}; p)$.

2. A trichotomy for sum-like equivalence relations

We denote the set of all non-negative real numbers by $\mathbb{R}^+$.

Definition 2.1. Let X_n, $n \in \mathbb{N}$ be a sequence of non-empty sets, $\psi_n : X_n^2 \to \mathbb{R}^+$. We define a relation $E((X_n, \psi_n)_{n\in\mathbb{N}})$ on $\prod_{n\in\mathbb{N}} X_n$ by

$$(x, y) \in E((X_n, \psi_n)_{n\in\mathbb{N}}) \iff \sum_{n\in\mathbb{N}} \psi_n(x(n), y(n)) < +\infty$$

for $x, y \in \prod_{n\in\mathbb{N}} X_n$. If $(X_n, \psi_n) = (X, \psi)$ for every $n \in \mathbb{N}$, we write $E(X, \psi) = E((X_n, \psi_n)_{n\in\mathbb{N}})$ for the sake of brevity.

It is hard to find a natural necessary and sufficient condition to determine when $E((X_n, \psi_n)_{n\in\mathbb{N}})$ is an equivalence relation. We need the following definition:

Definition 2.2. If $E((X_n, \psi_n)_{n\in\mathbb{N}})$ is an equivalence relation, we call it a *sum-like equivalence relation*. Furthermore, if X_n, $n \in \mathbb{N}$ is a sequence of Polish spaces and every ψ_n is Borel function, it is called a *sum-like Borel equivalence relation*.

The following easy lemma is very useful for the study of sum-like equivalence relations.

Lemma 2.3. *Let $E = E((X_n, \psi_n)_{n\in\mathbb{N}})$ be a sum-like equivalence relation. For $x, y \in \prod_{n\in\mathbb{N}} X_n$, we have*

$$(x, y) \in E \iff \sum_{x(n)\neq y(n)} \psi_n(x(n), y(n)) < +\infty.$$

Proof. Since E is an equivalence relation, $(x, x) \in E$. It follows that

$$\sum_{x(n)=y(n)} \psi_n(x(n), y(n)) \leq \sum_{n\in\mathbb{N}} \psi_n(x(n), x(n)) < +\infty.$$

Then the lemma follows. □

Definition 2.4. Let (X_n, d_n), $n \in \mathbb{N}$ be a sequence of pseudo-metric spaces. For $p \geq 1$, an *ℓ_p-like equivalence relation* $E((X_n, d_n)_{n\in\mathbb{N}}; p)$ is $E((X_n, \psi_n)_{n\in\mathbb{N}})$ where $\psi_n(u, v) = d_n(u, v)^p$ for $u, v \in X_n$. If $(X_n, d_n) = (X, d)$ for every $n \in \mathbb{N}$, we write $E(X, d; p) = E((X_n, d_n)_{n\in\mathbb{N}}; p)$ for the sake of brevity.

Proposition 2.5. *Let $E((X_n, d_n)_{n\in\mathbb{N}}; p)$ be an ℓ_p-like equivalence relation. Then there exists metric d'_n on X_n for each n such that $E((X_n, d'_n)_{n\in\mathbb{N}}; p) = E((X_n, d_n)_{n\in\mathbb{N}}; p)$.*

Proof. We can see that following d'_n's meet the requirements.

$$d'_n(u, v) = \begin{cases} 0, & u = v, \\ 2^{-n}, & u \neq v, d_n(u, v) \leq 2^{-n}, \\ d_n(u, v), & d_n(u, v) > 2^{-n}, \end{cases}$$

for $u, v \in X_n$. □

Proposition 2.6. *Let E_i, $i \in \mathbb{N}$ be a sequence of Borel equivalence relations. Then for every $p \geq 1$ there is an ℓ_p-like Borel equivalence relation $E = E((X_n, d_n)_{n\in\mathbb{N}}; p)$ such that $E_i \leq_B E$ for $n \in \mathbb{N}$.*

Proof. For $i \in \mathbb{N}$, let E_i be a Borel equivalence relation on a Polish space Y_i. We define $\chi_i : Y_i^2 \to \mathbb{R}^+$ by

$$\chi_i(u, v) = \begin{cases} 0, & (u, v) \in E_i, \\ 1, & (u, v) \notin E_i. \end{cases}$$

Now fix a bijection $\langle \cdot, \cdot \rangle : \mathbb{N}^2 \to \mathbb{N}$. For every $i, j \in \mathbb{N}$, if $n = \langle i, j \rangle$, we denote $X_n = Y_i$ and $d_n = \chi_i$. It is easy to verify that $E = E((X_n, d_n)_{n\in\mathbb{N}}; p)$ is an ℓ_p-like Borel equivalence relation. For $i \in \mathbb{N}$, define $\theta_i : Y_i \to \prod_{n\in\mathbb{N}} X_n$ by

$$\theta_i(u)(\langle k, j \rangle) = \begin{cases} u, & k = i, \\ a_k, & k \neq i, \end{cases}$$

where $a_k \in Y_k$ is independent of u. Clearly θ_i is a Borel reduction of E_i to E. □

Now suppose that $E = E((X_n, \psi_n)_{n\in\mathbb{N}})$ is a sum-like Borel equivalence relation. Its position in the $\leq_B$-structure of Borel equivalence relations is determined by the following condition:

($\ell 1$) *$\forall c > 0 \exists x, y \in \prod_{n\in\mathbb{N}} X_n$ such that $\exists m \forall n > m(\psi_n(x(n), y(n)) < c)$ and*

$$\sum_{n\in\mathbb{N}} \psi_n(x(n), y(n)) = +\infty.$$

Lemma 2.7. *If ($\ell 1$) holds, then $\mathbb{R}^{\mathbb{N}}/\ell_1 \leq_B E$.*

Proof. Firstly, we show a well known fact: $\mathbb{R}^{\mathbb{N}}/\ell_1 \leq_B [0,1]^{\mathbb{N}}/\ell_1$. Fix a bijection $\langle \cdot, \cdot \rangle' : \mathbb{N} \times \mathbb{Z} \to \mathbb{N}$. For $z \in \mathbb{R}^{\mathbb{N}}$, we define $\theta'(z) \in [0,1]^{\mathbb{N}}$ as

$$\theta'(z)(\langle m, k\rangle') = \begin{cases} 0, & z(m) < k, \\ z(m) - k, & k \leq z(m) < k+1, \\ 1, & k+1 \leq z(m). \end{cases}$$

Then θ' witness that $\mathbb{R}^{\mathbb{N}}/\ell_1 \leq_B [0,1]^{\mathbb{N}}/\ell_1$.

Secondly, we construct a reduction of $[0,1]^{\mathbb{N}}/\ell_1$ to E.

For $l \in \mathbb{N}$, by condition ($\ell 1$), there exist $x_l, y_l \in \prod_{n\in\mathbb{N}} X_n$ such that $\exists m \forall n > m(\psi_n(x_l(n), y_l(n)) < 2^{-l})$ and $\sum_{n\in\mathbb{N}} \psi_n(x_l(n), y_l(n)) = +\infty$. Then we can select two sequences of natural numbers $(i_l)_{l\in\mathbb{N}}, (j_l)_{l\in\mathbb{N}}$ satisfying that

(i) $i_l < j_l < i_{l+1}$ for $l \in \mathbb{N}$;
(ii) $\psi_n(x_l(n), y_l(n)) < 2^{-l}$ for $i_l \leq n \leq j_l$;
(iii) $1 \leq \sum_{i_l \leq n \leq j_l} \psi_n(x_l(n), y_l(n)) < 1 + 2^{-l}$.

Fix an element $a_n \in X_n$ for every $n \in \mathbb{N}$. For $z \in [0,1]^{\mathbb{N}}$, we define $\vartheta(z) \in \prod_{n\in\mathbb{N}} X_n$ by

$$\vartheta(z)(n) = \begin{cases} x_l(n), & i_l \leq n \leq j_l, \sum_{i_l \leq m \leq n} \psi_m(x_l(m), y_l(m)) \leq z(l); \\ y_l(n), & i_l \leq n \leq j_l, \sum_{i_l \leq m \leq n} \psi_m(x_l(m), y_l(m)) > z(l); \\ a_n, & \text{otherwise}. \end{cases}$$

Note that, for $z, w \in [0,1]^{\mathbb{N}}$ and $l \in \mathbb{N}$, we have

$$|z(l) - w(l)| - 2^{-l} < \sum_{\substack{i_l \leq n \leq j_l \\ \vartheta(z)(n) \neq \vartheta(w)(n)}} \psi_n(\vartheta(z)(n), \vartheta(w)(n)) < |z(l) - w(l)| + 2^{-l}.$$

Therefore, by Lemma 2.3,

$$\begin{aligned} (\vartheta(z), \vartheta(w)) \in E \quad &\iff \sum_{\vartheta(z)(n)\neq\vartheta(w)(n)} \psi_n(\vartheta(z)(n), \vartheta(w)(n)) < +\infty \\ &\iff \sum_{l\in\mathbb{N}} |z(l) - w(l)| < +\infty \\ &\iff z - w \in \ell_1. \end{aligned}$$

It follows that $[0,1]^{\mathbb{N}}/\ell_1 \leq_B E$. □

If ($\ell 1$) fails, then there exists $c > 0$ such that

$$\exists m \forall n > m(\psi_n(x(n), y(n)) < c) \Rightarrow \sum_{n\in\mathbb{N}} \psi_n(x(n), y(n)) < +\infty$$

for $x, y \in \prod_{n\in\mathbb{N}} X_n$. Now we denote

$$F_n = \{(u, v) \in X_n^2 : \psi_n(u, v) < c\}.$$

Lemma 2.8. *If ($\ell 1$) fails, let $c > 0$ and F_n, $n \in \mathbb{N}$ be defined as above. Then*

(1) *for $x, y \in \prod_{n\in\mathbb{N}} X_n$,*
$$(x, y) \in E \iff \exists m \forall n > m((x(n), y(n)) \in F_n);$$
(2) *there exists N_0 such that, for $n > N_0$, F_n is a Borel equivalence relation on X_n.*

Proof. Since ($\ell 1$) fails, clause (1) is trivial.

(2) Firstly, assume for contradiction that, there exist a strictly increasing sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ and $u_k \in X_{n_k}$ such that $\psi_{n_k}(u_k, u_k) \geq c$. Select an $x \in \prod_{n\in\mathbb{N}} X_n$ with $x(n_k) = u_k$ for $k \in \mathbb{N}$. Then we have $\sum_{n\in\mathbb{N}} \psi_n(x(n), x(n)) = +\infty$. This is impossible, since E is an equivalence relation. So there exists N_1 such that, for $n > N_1, u \in X_n$, we have $\psi_n(u, u) < c$, i.e. $(u, u) \in F_n$.

Secondly, assume for contradiction that, there exist a strictly increasing sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ and $u_k, v_k \in X_{n_k}$ such that $\psi_{n_k}(u_k, v_k) < c, \psi_{n_k}(v_k, u_k) \geq c$. Select $x, y \in \prod_{n\in\mathbb{N}} X_n$ such that $x(n_k) = u_k, y(n_k) = v_k$ for $k \in \mathbb{N}$ and $x(n) = y(n)$ for other n. We have $\psi_n(x(n), y(n)) < c$ for $n > N_1$. Since ($\ell 1$) fails, $\sum_{n\in\mathbb{N}} \psi_n(x(n), y(n)) < +\infty$, i.e. $(x, y) \in E$. Clearly $\sum_{n\in\mathbb{N}} \psi_n(y(n), x(n)) = +\infty$, so $(y, x) \notin E$. A contradiction! Hence there exists N_2 such that, for $n > N_2, u, v \in X_n$, we have $\psi_n(u, v) < c \Rightarrow \psi_n(v, u) < c$, i.e. $(u, v) \in F_n \Rightarrow (v, u) \in F_n$.

With a similar argument, we can prove that there exists N_3 such that, for $n > N_3, u, v, r \in X_n$, we have $(u, v), (v, r) \in F_n \Rightarrow (u, r) \in F_n$.

In summary, for $n > N_0 = \max\{N_1, N_2, N_3\}$, F_n is an equivalence relation. Since ψ_n is Borel, so is F_n. □

Recall that $E_0(\mathbb{N})$ is an equivalence relation on $\mathbb{N}^{\mathbb{N}}$ similar to E_0 on $2^{\mathbb{N}}$. For $x, y \in \mathbb{N}^{\mathbb{N}}$, $(x, y) \in E_0(\mathbb{N}) \Leftrightarrow \exists m \forall n > m(x(n) = y(n))$. It is well known that $E_0 \sim_B E_0(\mathbb{N})$ (see Proposition 6.1.2 of [4]).

Lemma 2.9. *For any equivalence relation E on $\prod_{n\in\mathbb{N}} X_n$, if there is a sequence $F_n \subseteq X_n^2$, $n \in \mathbb{N}$ such that (1) and (2) of Lemma 2.8 hold, then either $E_1 \leq_B E$, $E \sim_B E_0$, or E is trivial, i.e. all elements in $\prod_{n\in\mathbb{N}} X_n$ are equivalent.*

Proof. For $n > N_0$, since F_n is a Borel equivalence relation on X_n, from the Silver dichotomy theorem [13], either there are at most countably many F_n-equivalence classes or there are perfectly many F_n-equivalence classes.

Case 1. There exists a strictly increasing sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ such that there are perfectly many F_{n_k}-equivalence classes. Then there is a continuous embedding $h_k : 2^{\mathbb{N}} \to X_{n_k}$ for every $k \in \mathbb{N}$ such that $(h_k(z), h_k(w)) \in F_{n_k}$ iff $z = w$. Define $\theta : 2^{\mathbb{N}\times\mathbb{N}} \to \prod_{n\in\mathbb{N}} X_n$ by

$$\theta(x)(n) = \begin{cases} h_k(x(k,\cdot)), & n = n_k, \\ a_n, & \text{otherwise}, \end{cases}$$

where $a_n \in X_n$ is independent of x. By Lemma 2.8.(1), it is straightforward to check that θ is a reduction of E_1 to E.

Case 2. There exists N such that, for $n > N$, F_n has only one equivalence class. From Lemma 2.8.(1), we see that E is trivial.

Case 3. If case 1 fails, then there exists N' such that, for $n > N'$, F_n has at most countably many equivalence classes. So $E \le_B E_0(\mathbb{N})$. If case 2 fails, then there exists a strictly increasing sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ such that F_{n_k} has more than one equivalence class. Thus $E_0 \le_B E$. Since $E_0 \sim_B E_0(\mathbb{N})$, we have $E \sim_B E_0$. □

Now we have already completed the proof of the following trichotomy.

Theorem 2.10. *Let $E = E((X_n, \psi_n)_{n\in\mathbb{N}})$ be a sum-like Borel equivalence relation. Then either $\mathbb{R}^{\mathbb{N}}/\ell_1 \le_B E$, $E_1 \le_B E$, or $E \le_B E_0$.*

Corollary 2.11. *Let $E = E((X_n, \psi_n)_{n\in\mathbb{N}})$ be a sum-like Borel equivalence relation. If $E \le_B \mathbb{R}^{\mathbb{N}}/\ell_1$, then either $E \le_B E_0$ or $E \sim_B \mathbb{R}^{\mathbb{N}}/\ell_1$.*

Proof. It is well known that $E_1 \not\le_B \mathbb{R}^{\mathbb{N}}/\ell_1$ (see [9] Theorem 4.2), so if $E \le_B \mathbb{R}^{\mathbb{N}}/\ell_1$, then $E_1 \not\le_B E$. Hence the corollary follows. □

It was shown by R. Dougherty and G. Hjorth [2] that, for $p, q \ge 1$,

$$\mathbb{R}^{\mathbb{N}}/\ell_p \le_B \mathbb{R}^{\mathbb{N}}/\ell_q \iff p \le q.$$

The following corollary shows that, for $p \le 1$, the situation is different.

Corollary 2.12. *For $0 < p \le 1$, we have $\mathbb{R}^{\mathbb{N}}/\ell_p \sim_B \mathbb{R}^{\mathbb{N}}/\ell_1$.*

Proof. Note that $\mathbb{R}^{\mathbb{N}}/\ell_p = E(\mathbb{R}, \psi)$ where $\psi(u, v) = |u - v|^p$ for $u, v \in \mathbb{R}$. It is easy to see that $(\ell 1)$ holds for $E(\mathbb{R}, \psi)$, so $\mathbb{R}^{\mathbb{N}}/\ell_1 \le_B \mathbb{R}^{\mathbb{N}}/\ell_p$.

For the other direction, we claim that $\mathbb{R}^{\mathbb{N}}/\ell_p \le_B \mathbb{R}^{\mathbb{N}}/\ell_q$ for $0 < p < q \le 1$.

We sketch the proof for Theorem 1.1 of [2], that $\mathbb{R}^{\mathbb{N}}/\ell_p \le_B \mathbb{R}^{\mathbb{N}}/\ell_q$ for $1 \le p < q$, and check that it is also valid for $0 < p < q \le 1$.

It will suffice to prove for the case $0 < \frac{q}{2} < p < q \le 1$. We denote $\rho = \frac{p}{q}$ and $r = 4^{-\rho}$. Then $\frac{1}{2} < \rho < 1$ and $\frac{1}{4} < r < \frac{1}{2}$. In [2] p. 1838, the authors constructed a continuous function $\bar{K}_r : \mathbb{R} \to \mathbb{R}^2$, and proved that there are $m', M' > 0$ such that

$$m'|s - t|^\rho \le \|\bar{K}_r(s) - \bar{K}_r(t)\|_2 \le M'|s - t|^\rho,$$

for $s, t \in [i - 1, i + 1], i \in \mathbb{Z}$. And $\|\bar{K}_r(s) - \bar{K}_r(t)\|_2 \ge 1$ if s, t are not in the same interval $[i - 1, i + 1], i \in \mathbb{Z}$.

Define a mapping $\theta : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ such that, for $x \in \mathbb{R}^{\mathbb{N}}$ and $k \in \mathbb{N}$,

$$\bar{K}_r(x(k)) = (\theta(x)(2k), \theta(x)(2k+1)).$$

For $w = (s,t) \in \mathbb{R}^2$, denote $\|w\|_q = (|s|^q + |t|^q)^{\frac{1}{q}}$. Note that

$$\frac{1}{\sqrt{2}}\|w\|_2 \leq \|w\|_\infty \leq \|w\|_q \leq 2^{\frac{1}{q}}\|w\|_\infty \leq 2^{\frac{1}{q}}\|w\|_2.$$

For $x, y \in \mathbb{R}^{\mathbb{N}}$, we have

$$\begin{aligned} \theta(x) - \theta(y) \in \ell_q \quad &\iff \textstyle\sum_{k\in\mathbb{N}} \|\bar{K}_r(x(k)) - \bar{K}_r(y(k))\|_q^q < +\infty \\ &\iff \textstyle\sum_{k\in\mathbb{N}} \|\bar{K}_r(x(k)) - \bar{K}_r(y(k))\|_2^q < +\infty \\ &\iff \textstyle\sum_{k\in\mathbb{N}} |x(k) - y(k)|^p < +\infty \\ &\iff x - y \in \ell_p. \end{aligned}$$

Thus, θ is a reduction of $\mathbb{R}^{\mathbb{N}}/\ell_p$ to $\mathbb{R}^{\mathbb{N}}/\ell_q$. □

3. Metrization

In this section, we show that a sum-like equivalence relation $E((X_n, \psi_n)_{n\in\mathbb{N}})$ coincides with an ℓ_p-like equivalence relation if the following conditions hold.

($m1$) *Denote $X = \bigcup_{n\in\mathbb{N}} X_n$. There is a unique function $\psi : X^2 \to \mathbb{R}^+$ such that $\psi_n = \psi \restriction X_n^2$ and $\psi(u,v) = 1$ if no X_n contains both u, v.*

($m2$) *For any $u, v, r \in X$, if $u, v, r \in X_n$, then there exists $m > n$ such that $u, v, r \in X_m$.*

Let $\overline{\psi}_n = \min\{\psi_n, 1\}$. We can see that $E((X_n, \overline{\psi}_n)_{n\in\mathbb{N}}) = E((X_n, \psi_n)_{n\in\mathbb{N}})$. Therefore, we may assume $\psi_n \leq 1$ if needed.

Lemma 3.1. *Let $(X_n, \psi_n)_{n\in\mathbb{N}}$ satisfy ($m1$) and ($m2$). If $\psi_n \leq 1$, then $E((X_n, \psi_n)_{n\in\mathbb{N}})$ is an equivalence relation iff the following conditions hold:*

(i) *$\psi(u,u) = 0$ for $u \in X$;*

(ii) *there is a $C \geq 1$ such that for $u, v, r \in X$,*

$$\psi(v,u) \leq C\psi(u,v); \quad \psi(u,r) \leq C(\psi(u,v) + \psi(v,r)).$$

Proof. If conditions (i) and (ii) hold, it is trivial that $E((X_n, \psi_n)_{n\in\mathbb{N}})$ is an equivalence relation. We only need to prove the other direction.

Now assume that $E((X_n, \psi_n)_{n\in\mathbb{N}})$ is an equivalence relation.

Firstly, for any $u \in X$, by ($m2$), there is an infinite set $I \subseteq \mathbb{N}$ such that $u \in X_n$ for $n \in I$. Let $x \in \prod_{n\in\mathbb{N}} X_n$ with $x(n) = u$ for $n \in I$. Since $(x,x) \in E((X_n, \psi_n)_{n\in\mathbb{N}})$, we have $\psi(u,u) = 0$.

Secondly, for $u, v \in X$, if $\psi(u,v) = 0$, we claim that $\psi(v,u) = 0$. By ($m2$), there is an infinite set $I \subseteq \mathbb{N}$ such that $u, v \in X_n$ for $n \in I$. Therefore, for any $x, y \in \prod_{n\in\mathbb{N}} X_n$, if $x(n) = u, y(n) = v$ for $n \in I$ and $x(n) = y(n)$ for $n \notin I$, we have $(x,y) \in E((X_n, \psi_n)_{n\in\mathbb{N}})$. Hence $(y,x) \in E((X_n, \psi_n)_{n\in\mathbb{N}})$. It follows that $\sum_{n\in I} \psi(v,u) = \sum_{n\in I} \psi(y(n), x(n)) < +\infty$. Thus $\psi(v,u) = 0$.

Now assume for contradiction that, for every $k \in \mathbb{N}$ there are $u_k, v_k \in X$ such that $\psi(v_k, u_k) > 2^k \psi(u_k, v_k) > 0$. Since $\psi_n \leq 1$, $0 < \psi(u_k, v_k) < 2^{-k}$. By ($m2$), there are infinitely many n such that $u_k, v_k \in X_n$.

Select a finite set $I_k \subseteq \mathbb{N}$ for every k satisfying that

(i) $u_k, v_k \in X_n$ for $n \in I_k$;
(ii) $2^{-k} \leq |I_k|\psi(u_k, v_k) \leq 2^{-(k-1)}$;
(iii) if $k_1 < k_2$, then $\max I_{k_1} < \min I_{k_2}$.

Now we define $x, y \in \prod_{n\in\mathbb{N}} X_n$ by

$$\begin{cases} x(n) = u_k, y(n) = v_k, & n \in I_k, k \in \mathbb{N}, \\ x(n) = y(n) = a_n, & \text{otherwise,} \end{cases}$$

where $a_n \in X_n$ is independent of x and y. Then we have

$$\sum_{n\in\mathbb{N}} \psi(x(n), y(n)) = \sum_{k\in\mathbb{N}} |I_k|\psi(u_k, v_k) \leq \sum_{k\in\mathbb{N}} 2^{-(k-1)} < +\infty,$$

so $(x, y) \in E((X_n, \psi_n)_{n\in\mathbb{N}})$. On the other hand, we have

$$\sum_{n\in\mathbb{N}} \psi(y(n), x(n)) = \sum_{k\in\mathbb{N}} |I_k|\psi(v_k, u_k) > \sum_{k\in\mathbb{N}} 2^k |I_k|\psi(u_k, v_k) \geq \sum_{k\in\mathbb{N}} 1 = +\infty,$$

so $(y, x) \notin E((X_n, \psi_n)_{n\in\mathbb{N}})$. A Contradiction! We complete the proof of that there is $C_1 \geq 1$ such that $\psi(u, v) \leq C_1\psi(v, u)$ for $u, v \in X$.

With a similar argument, we can prove that there is $C_2 \geq 1$ such that $\psi(u, r) \leq C_2(\psi(u, v) + \psi(v, r))$ for $u, v, r \in X$. □

Lemma 3.2. *Let $E((X_n, \psi_n)_{n\in\mathbb{N}}), E((X_n, \varphi_n)_{n\in\mathbb{N}})$ be two sum-like equivalence relations, both satisfying* (m1) *and* (m2)*. If $\varphi_n \leq 1$, then*

$$\begin{aligned} & E((X_n, \psi_n)_{n\in\mathbb{N}}) \subseteq E((X_n, \varphi_n)_{n\in\mathbb{N}}) \\ \iff & \exists A \geq 1 \forall u, v \in X(\varphi(u, v) \leq A\psi(u, v)). \end{aligned}$$

Proof. "⇐" is trivial. "⇒" follows similarly as the proof of Lemma 3.1. □

Corollary 3.3. *Let $E((X_n, \psi_n)_{n\in\mathbb{N}}), E((X_n, \varphi_n)_{n\in\mathbb{N}})$ be two sum-like equivalence relations, both satisfying* (m1) *and* (m2)*. If $\psi_n, \varphi_n \leq 1$, then*

$$\begin{aligned} & E((X_n, \psi_n)_{n\in\mathbb{N}}) = E((X_n, \varphi_n)_{n\in\mathbb{N}}) \\ \iff & \exists A \geq 1 \forall u, v \in X(A^{-1}\psi(u, v) \leq \varphi(u, v) \leq A\psi(u, v)). \end{aligned}$$

Before introducing the Metrization lemma, we recall several basic notions on relations. Let X be a non-empty set, we denote $\Delta(X) = \{(u, u) : u \in X\}$. A subset $U \subseteq X^2$ is called symmetric if $(u, v) \in U \Rightarrow (v, u) \in U$ for $u, v \in X$. For $U, V \subseteq X^2$ we define $U \circ V \subseteq X^2$ by

$$(u, r) \in U \circ V \iff \exists v((u, v) \in U, (v, r) \in V) \quad (\forall u, v, r \in X).$$

Lemma 3.4 (Metrization lemma [10], p. 185)**.** *Let U_n, $n \in \mathbb{N}$ be a sequence of subsets of X^2 such that*

(i) *$U_0 = X^2$;*
(ii) *each U_n is symmetric and $\Delta(X) \subseteq U_n$;*
(iii) *$U_{n+1} \circ U_{n+1} \circ U_{n+1} \subseteq U_n$ for each n.*

Then there is a pseudo-metric d on X satisfying that

$$U_n \subseteq \{(u, v) : d(u, v) < 2^{-n}\} \subseteq U_{n-1} \quad (\forall n \geq 1).$$

Theorem 3.5. *Let $(X_n,\psi_n)_{n\in\mathbb{N}}$ satisfy $(m1)$ and $(m2)$. Then $E((X_n,\psi_n)_{n\in\mathbb{N}})$ is an equivalence relation iff it is an ℓ_p-like equivalence relation.*

Proof. Assume that $E((X_n,\psi_n)_{n\in\mathbb{N}})$ is an equivalence relation. Without loss of generality, we may assume that $\psi_n\leq 1$. From Lemma 3.1, there is $C\geq 1$ such that, for $u,v,r\in X$,

$$\psi(v,u)\leq C\psi(u,v) \text{ and } \psi(u,r)\leq C(\psi(u,v)+\psi(v,r)).$$

Therefore, if $\psi(u,v),\psi(v,r),\psi(r,s)<\varepsilon$, then

$$\psi(u,s)\leq C(\psi(u,v)+C(\psi(v,r)+\psi(r,s)))<(2C^2+C)\varepsilon.$$

Now denote $B=2C^2+C$. We define $U_0=X^2$ and

$$U_n=\{(u,v):\psi(u,v)<B^{-n},\psi(v,u)<B^{-n}\}\quad(n\geq 1).$$

It follows that, for each n, U_n is symmetric and $U_{n+1}\circ U_{n+1}\circ U_{n+1}\subseteq U_n$. By Lemma 3.1.(i), $\Delta(X)\subseteq U_n$. Then the metrization lemma gives a pseudo-metric d on X such that $U_n\subseteq\{(u,v):d(u,v)<2^{-n}\}\subseteq U_{n-1}$.

It is easy to check that, $d(u,v)=0$ iff $\psi(u,v)=0$. If $\psi(u,v)\geq B^{-2}$, then $(u,v)\notin U_2$, $d(u,v)\geq 2^{-3}$.

Denote $p=\log_2 B\geq 1$. If $0<\psi(u,v)<B^{-2}$, assume that $B^{-(n+1)}\leq\psi(u,v)<B^{-n}$ for some $n\geq 2$. Then $\psi(v,u)<CB^{-n}<B^{-(n-1)}$. It follows that $(u,v)\in U_{n-1},(u,v)\notin U_{n+1}$, $2^{-(n+2)}\leq d(u,v)<2^{-(n-1)}$. So

$$B^{-2}d(u,v)^p<B^{-2}(2^{-(n-1)})^p\leq\psi(u,v)<B^2(2^{-(n+2)})^p\leq B^2d(u,v)^p.$$

Therefore, we have $E((X_n,\psi_n)_{n\in\mathbb{N}})=E((X_n,d\restriction X_n);p)$. □

Corollary 3.6. *Let $(X_n,\psi_n)_{n\in\mathbb{N}}$ satisfy that, for $n<m$, $X_n\subseteq X_m$ and $\psi_n=\psi_m\restriction X_n^2$. Then $E((X_n,\psi_n)_{n\in\mathbb{N}})$ is an equivalence relation iff it is an ℓ_p-like equivalence relation.*

In particular, $E(X,\psi)$ is an equivalence relation iff there are pseudo-metric d on X and $p\geq 1$ such that $E(X,\psi)=E(X,d;p)$.

4. Further remarks

Let us consider a special case of sum-like equivalence relation. Let $\psi_f(u,v)=f(|u-v|)$ where $f:\mathbb{R}^+\to\mathbb{R}^+$. Then conditions (i) and (ii) for $E(\mathbb{R},\psi_f)$ in Lemma 3.1 read as (see also [11], Proposition 2).

(i) $f(0)=0$;

(ii) *there is a $C\geq 1$ such that for $s,t\in\mathbb{R}^+$,*

$$f(s+t)\leq C(f(s)+f(t)),\quad f(s)\leq C(f(s+t)+f(t)).$$

Denote $\mathcal{N}_f=\{x\in\mathbb{R}^{\mathbb{N}}:\sum_{n\in\mathbb{N}}f(|x(n)|)<+\infty\}$. Then $E(\mathbb{R},\psi_f)$ is an equivalence relation iff $\mathcal{N}_f$ is a subgroup of $(\mathbb{R}^{\mathbb{N}},+)$. Furthermore, another interesting problem is to determine when $\mathcal{N}_f$ is a linear subspace of $\mathbb{R}^{\mathbb{N}}$. This problem was studied by S. Mazur and W. Orlicz (see [12], 1.7). It was also considered in [12] that when $\mathcal{N}_f$'s are Banach spaces.

Theorem 4.1 (Mazur-Orlicz)**.** *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy that, as $n \to \infty$, $f(t_n) \to 0$ iff $t_n \to 0$. The necessary and sufficient condition for $\mathcal{N}_f$ to be a linear space is that*

(a) *there exist constants $C > 0, \varepsilon > 0$ such that $f(s+t) \leq C(f(s)+f(t))$ for $s, t < \varepsilon$;*
(b) *for every $\rho > 0$ there are constants $D > 0, \delta > 0$ such that $f(s) \leq Df(t)$ for $t < \delta, s < \rho t$.*

Note that for $t_n \geq \min\{\varepsilon, \delta\} > 0$, $n \in \mathbb{N}$, we have $f(t_n) \not\to 0$. Thus there is $c > 0$ such that $f(t) \geq c$ for $t \geq \min\{\varepsilon, \delta\}$. If we assume that $f \leq 1$, then conditions (a) and (b) in this theorem turn to

(a)' *there exists a constant $C' > 0$ such that $f(2s) \leq C' f(s)$ for $s \in \mathbb{R}^+$;*
(b)' *there exists a constant $D' > 0$ such that $f(s) \leq D' f(t)$ for $s < t$.*

For almost all known examples of sum-like equivalence relations $E(\mathbb{R}, \psi_f)$, $\mathcal{N}_f$'s are linear spaces. In the end, we present an example in which $\mathcal{N}_f$ is not linear as follows.

Example 4.2. Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function with $g(0) = 0$, such that $g'(t)$ is decreasing with $\lim_{t\to 0} g'(t) = +\infty$. For example, $g(x) = \sqrt{x}$ is such a function. Let $(a_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence of positive numbers with $\lim_{n\to\infty} a_n = 0$.

Denote $k_n = \frac{g(a_n)}{a_n}$. Then $k_n < k_{n+1}$. We consider equations $y = k_n x$ and $y - g(a_{n+1}) = -k_{n+1}(x - a_{n+1})$. Their solution is $(b_n, k_n b_n)$ where $b_n = \frac{2k_{n+1}a_{n+1}}{k_n + k_{n+1}}$. We can see that $a_{n+1} < b_n < a_n$.

Now define $f : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$f(t) = \begin{cases} 0, & t = 0, \\ -k_{n+1}(t - a_{n+1}) + g(a_{n+1}), & a_{n+1} \leq t < b_n, \\ k_n t, & b_n \leq t < a_n, \\ g(a_0), & a_0 \leq t. \end{cases}$$

It is easy to check that, for $s, t \in \mathbb{R}^+$,

$$f(s+t) \leq f(s) + f(t), \quad f(s) \leq f(s+t) + f(t).$$

Note that $\frac{f(a_{n+1})}{f(b_n)} = \frac{1}{2}\left(1 + \frac{k_{n+1}}{k_n}\right)$. Since $\lim_{t\to 0} \frac{g(t)}{t} = \lim_{t\to 0} g'(t) = +\infty$, we can find a sequence $(a_n)_{n\in\mathbb{N}}$ such that $\frac{k_{n+1}}{k_n} \to +\infty$, $\frac{f(a_{n+1})}{f(b_n)} \to +\infty$ as $n \to \infty$. Then condition (b) in Theorem 4.1 fails.

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